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Symmetric Duality, and a Convergent Subgradient Method for Discrete, Linear, Constrained Approximation Problems with Arbitrary Norms Appearing in the Objective Function and in the Constraints*

W. OETTLI

Universität Mannheim, Mannheim, Germany

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A method is described for solving certain dual pairs of constrained approximation problems.

1

In what follows we describe a subgradient method for certain discrete, linear, constrained approximation problems. The method is based on duality results which we include here for the sake of completeness. The geometric idea underlying the method is the same as the one described in [1] for the case of linear programming problems. There is, therefore, a certain overlap between these two articles. We found it, however, worthwhile to treat the approximation problem (and the corresponding duality theory) in its own right. We should note perhaps that Krabs [2] has advocated a so-called pseudogradient method for solving problems of Chebyshev-approximation. The additional conditions required by his method to ensure convergence seem, however, difficult to ascertain a priori. The method we propose always converges, provided the problems under consideration have optimal solutions at all (not necessarily unique ones).

2

Suppose we are given three systems of functions $\{f_j(t)\}$, $\{g_j(t)\}$, $\{h_j(t)\}$ $(j = 0, 1,..., m)$, all defined on a finite point set $\{t_1 ,..., t_k\}$. Consider the problem of finding a linear combination $\sum_1^m x_j f_j$ whose distance to f_0 is

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minimum, while the distance of the corresponding linear combination $\sum_1^m x_j g_j$ to g_0 is below a certain threshold, and the values of $\sum_1^m x_j h_j$ are above the corresponding values of h_0. If we identify the values $f_j(t_l)$ with the k-vector f_j, and if we measure the distance from f_0 and g_0 by means of norms π and ρ in $\mathbb{R}^k$, then the above problem is the problem of

minimizing

$$\pi\left(f_0 - \sum_1^m x_j f_j\right),$$

subject to

$$\rho\left(g_0 - \sum_1^m x_j g_j\right) \leqslant 1,$$

$$h_0 - \sum_1^m x_j h_j \leqslant 0.$$

Introducing additional variables and noting that a given equality can always be expressed by two inequalities and that an arbitrary variable can be represented as the difference of two nonnegative variables, we see that the above problem is a special case of the following problem P.

$$P\colon \min\{F(x, y) \mid (x, y) \in S\},$$

where $F(x, y) = \pi(Cx + Dy + \beta) + a^T x + b^T y$ and S is given by the conditions

$$Ax + By + \alpha \leqslant 0,$$

$$x \geqslant 0, \qquad \rho(y) \leqslant 1.$$

Here x, y are vectors of real-valued variables; A, B, C, D denote matrices of appropriate size, π and ρ are given norms in the appropriate spaces.

We say that (x, y) is *feasible* for P, if $(x, y) \in S$, *strictly feasible*, if $(x, y) \in S$ and $\rho(y) < 1$.

In analogy with the duality theory for linear programs [3, 4] we may formulate for the given *primal* minimization problem P a *dual* maximization problem D,

$$D\colon \max\{G(\xi, \eta) \mid (\xi, \eta) \in \Sigma\},$$

where $G(\xi, \eta) = -\rho^*(-B^T\xi - D^T\eta - b) + \alpha^T\xi + \beta^T\eta$ and Σ is given by the conditions

$$A^T\xi + C^T\eta + a \geqslant 0,$$

$$\xi \geqslant 0, \qquad \pi^*(\eta) \leqslant 1.$$

Here T denotes transposition, and $*$ denotes the conjugate norm,

$$\pi^*(\eta) = \sup\{\eta^T z \mid \pi(z) \leqslant 1\}.$$

(We shall not need the symmetry of the norms π and ρ. Note that $\pi^{**} = \pi$.) We see that D has exactly the same structure as P, i.e., the pair P, D is symmetric. P and D are linked by the following theorem.

DUALITY AND EXISTENCE THEOREM. *If both problems P and D have strictly feasible points, or one has strictly feasible points and no linear part in the objective function, then both have optimal solutions, and a necessary and sufficient condition for optimal solutions is that*

$$(x, y) \in S, \qquad (\xi, \eta) \in \Sigma, \qquad F(x, y) \leqslant G(\xi, \eta). \tag{1}$$

For the sake of completeness, and since we are not able to give a precise reference, we supply a proof, based on a theorem of A. Ghouila-Houri, in the appendix.

3

We suppose that the conditions of the above theorem are satisfied, so that P and D have optimal solutions, and these can be found by solving the system (1). By combining all variables into $z = (x, y, \xi, \eta) \in \mathbb{R}^m$ we may write (1) in compact form as

$$f_j(z) \leqslant 0 \qquad (j = 1,..., N), \tag{2}$$

where the functions f_j are convex. We denote by Z the nonempty solution set of (2). Let $p(\cdot)$ be a monotonic norm in $\mathbb{R}^N$, and

$$\varphi(z) = p(f_1^+(z),..., f_N^+(z)), \qquad f_j^+(z) = \max\{0, f_j(z)\}.$$

Then φ is easily seen to be a convex function [1], and Z may equally well be described as the set of points giving $\varphi(z)$ the value zero, which, incidentally, is the minimal value of $\varphi(z)$ over $\mathbb{R}^m$. Our original problem of solving P and D thus has been transformed into the problem of minimizing the convex function $\varphi(z)$ over $\mathbb{R}^m$. φ is not differentiable everywhere; therefore ordinary gradient methods for minimizing φ may fail. However, φ, being a convex function, has in each point z at least one *subgradient*. This is a vector t such that the support inequality

$$(\zeta - z)^T t \leqslant \varphi(\zeta) - \varphi(z)$$

is valid for all $\zeta \in \mathbb{R}^m$. The set of all subgradients in z is denoted by $\partial\varphi(z)$[1].

[1] If φ is differentiable, the subgradient is unique and equals the gradient. Subgradients of φ may be calculated, if the subgradients of f_j and p are available. On bounded subsets of $\mathbb{R}^m$, the subgradients of φ are also bounded [1]. Note that $0 \notin \partial\varphi(z)$ if $z \notin Z$.

We may then use a subgradient method for minimizing φ. The fact that the minimal value of φ, zero, is known a priori will enable us to give an explicit prescription for the step-length.

4

For finding an element of Z the following subgradient method may be used: Starting with an arbitrary point $z^0 \in \mathbb{R}^m$ we define for $z^\nu \notin Z$,

$$z^{\nu+1} = z^\nu - \lambda_\nu \cdot \frac{\varphi(z^\nu)}{|t_\nu|^2} t_\nu ,$$

$$t_\nu \in \partial\varphi(z^\nu), \qquad \lambda_\nu \in (0, 2), \tag{3}$$

$$\sum_0^\infty \sigma_\nu = +\infty,$$

where $\sigma_\nu = \lambda_\nu(2 - \lambda_\nu)$. We assume that $z^\nu \notin Z$ for all ν. Then z^ν converges to an element $\hat{z} \in Z$.

Proof of convergence. First we note that

$$|z^{\nu+1} - z|^2 \leqslant |z^\nu - z|^2 - \sigma_\nu \frac{\varphi^2(z^\nu)}{|t_\nu|^2} \; \forall z \in Z, \tag{4}$$

with $|\cdot|$ denoting Euclidian distance. This follows by squaring out the left-hand side, after substitution for $z^{\nu+1}$ from (3), and by using the support inequality, which, because of $\varphi(z) = 0$, reads

$$(z - z^\nu)^T t_\nu \leqslant -\varphi(z^\nu). \tag{5}$$

Equation (4) implies that all iterates z^ν are bounded. Consequently

$$|t_\nu| \leqslant M \qquad \text{for all } \nu. \tag{6}$$

If we had $\varphi(z^\nu) \geqslant \alpha > 0$ for all ν, then from (4) and (6) by summing up we would have for all K,

$$\frac{\alpha^2}{M^2} \sum_{\nu=0}^{K-1} \sigma_\nu \leqslant |z^0 - z|^2 - |z^K - z|^2,$$

a contradiction, since the left-hand side tends to $+\infty$, whereas the right-hand side is bounded by $|z^0 - z|^2$. Therefore there exists a subsequence $z^{\bar{\nu}}$ such that $\varphi(z^{\bar{\nu}}) \to 0$, and even $z^{\bar{\nu}} \to \hat{z}$ (because of boundedness). It follows that $\varphi(\hat{z}) = 0$, whence $\hat{z} \in Z$. But then it follows from (4) that $|z^\nu - \hat{z}|$ is monotonically decreasing for the whole sequence. Therefore the whole sequence convergences to $\hat{z}$. Q.E.D.

The recursion (3) in itself is by no means new. For $m = 1$ (the one-dimensional case) it reduces to a Newton-step for finding a root of $\varphi(z) = 0$. An early example of its use for general m can be found in [5] (for systems of equations), a more recent one in [6].

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We make now the additional assumption that the norms π and ρ have the property of being the maximum of a *finite* family of linear forms. Then π^* and ρ^* have the same property, too, and this implies that all the functions f_j appearing in (2) have the property of being the maximum of a finite family of linear functions; thus

$$f_j(z) = \max_{i \in H_j} l_i(z), \tag{7}$$

with l_i linear, H_j finite. We also make the additional assumption that Z is a singleton,

$$Z = \{\hat{z}\}.$$

Under these two additional assumptions we have

$$\frac{\varphi(z)}{| z - \hat{z} |} \geqslant m > 0 \qquad \text{for all} \quad z \neq \hat{z}. \tag{8}$$

Proof of (8). $\hat{z}$ is then the unique solution of the system

$$l_i(z) \leqslant 0, \qquad i \in \bigcup_j H_j, \tag{9}$$

which under (7) is equivalent to (2). Let $\hat{H}_j = \{i \in H_j \mid l_i(\hat{z}) = 0\}$. Then $\hat{z}$ is still the unique solution of (9), if we replace $\bigcup_j H_j$ by $\bigcup_j \hat{H}_j$. Define

$$\tilde{f}_j(z) = \max_{i \in \hat{H}_j} l_i(z) (= 0, \text{ if } \hat{H}_j \text{ empty}),$$

$$\tilde{\varphi}(z) = p[(\tilde{f}_1(z))^+, \ldots, (\tilde{f}_N(z))^+].$$

Then

$\tilde{\varphi}(z) > 0$ for $z \neq \hat{z}$ (otherwise z would solve (9)),

$\tilde{\varphi}(z) \leqslant \varphi(z)$ for all z (because of the monotonicity of p),

$\tilde{\varphi}(\hat{z} + \lambda t) = \lambda \tilde{\varphi}(\hat{z} + t)\ \forall \lambda \geqslant 0$ (homogeneity about $\hat{z}$).

Thus for all $z \neq \hat{z}$

$$\frac{\varphi(z)}{| z - \hat{z} |} \geqslant \min_{|\zeta - \hat{z}| = 1} \tilde{\varphi}(\zeta) = m > 0. \qquad \text{Q.E.D.}$$

From (8) we draw two important numerical conclusions: (a) From (5) follows $\varphi(z^\nu) \leqslant |z^\nu - \hat{z}| \cdot |t_\nu|$, and by (8)

$$|t_\nu| \geqslant \frac{\varphi(z^\nu)}{|z^\nu - \hat{z}|} \geqslant m > 0.$$

Thus the denominator appearing in (3) is bounded away from zero, and the method is *stable*. (b) From (4), (6) and (8) follows

$$\frac{|z^{\nu+1} - \hat{z}|}{|z^\nu - \hat{z}|} \leqslant \left[1 - \sigma_\nu \frac{\varphi^2(z^\nu)}{|z^\nu - \hat{z}|^2 \cdot |t_\nu|^2}\right]^{1/2}$$

$$\leqslant 1 - \frac{\sigma_\nu m^2}{2M^2}.$$

Thus the method converges at a *linear rate*, if σ_ν is chosen to be constant.

Appendix

In order to prove the duality and existence theorem we write

P: $\min F = \pi(z) + a^T x + b^T y$
subject to

$$Ax + By + \alpha \leqslant 0$$
$$-z + Cx + Dy + \beta = 0$$
$$x \geqslant 0, \qquad \rho(y) \leqslant 1$$

D: $\max G = -\rho^*(-\zeta) + \alpha^T \xi + \beta^T \eta$
subject to

$$A^T\xi + C^T\eta + a \geqslant 0$$
$$-\zeta + B^T\xi + D^T\eta + b = 0$$
$$\xi \geqslant 0, \qquad \pi^*(\eta) \leqslant 1.$$

(a) We note first that $F \geqslant G$ for any feasible points of P and D. Indeed, using in turn the norm constraints, the equality constraints, and the linear inequality constraints, we obtain

$$\begin{aligned} F(x, y, z) - G(\xi, \eta, \zeta) &= \pi(z) + a^T x + b^T y + \rho^*(-\zeta) - \alpha^T\xi - \beta^T\eta \\ &\geqslant \eta^T z + a^T x + b^T y - y^T\zeta - \alpha^T\xi - \beta^T\eta \\ &= -\xi^T(Ax + By + \alpha) + x^T(A^T\xi + C^T\eta + a) \\ &\geqslant 0. \end{aligned}$$

(b) Let now P and D have strictly feasible points. Then, in view of (a), the infimum of F over all feasible points of P, call it $\hat{F}$, is finite. Since P has strictly feasible points, but has no feasible points with $F(x, y, z) < \hat{F}$, a sharpened version of the Farkas–Minkowski lemma, first proved by

A. Ghouila-Houri [7, p. 66][2], gives the existence of "multipliers" $\xi \geqslant 0$, η, $\mu \geqslant 0$, $\lambda \geqslant 0$ such that

$$\hat{F} \leqslant \pi(z) + a^T x + b^T y + \xi^T(Ax + By + \alpha) + \eta^T(-z + Cx + Dy + \beta) \\ - \mu^T x + \lambda \cdot (\rho(y) - 1), \qquad \forall x, \forall y, \forall z.$$

Introducing $\zeta = B^T\xi + D^T\eta + b$ this gives

$$\hat{F} \leqslant [x^T(A^T\xi + C^T\eta + a - \mu)] + [\pi(z) - z^T\eta] + [\lambda\rho(y) + y^T\zeta] \\ - \lambda + \alpha^T\xi + \beta^T\eta, \qquad \forall x, \forall y, \forall z. \tag{10}$$

Since the bracketted expressions $[\cdots]$ appearing in (10) are positively homogeneous of order 1, inequality (10) can hold for all values of x, y, z only if the brackets are nonnegative for all values of x, y, z. The nonnegativity of the three brackets furnishes in turn

$$A^T\xi + C^T\eta + a - \mu = 0 \qquad (\text{i.e., } A^T\xi + C^T\eta + a \geqslant 0),$$
$$\pi^*(\eta) \leqslant 1, \qquad \rho^*(-\zeta) \leqslant \lambda.$$

Thus (ξ, η, ζ) is feasible for D. Inserting the last inequality in (10) and setting $x = y = z = 0$ we obtain from (10)

$$\hat{F} \leqslant -\rho^*(-\zeta) + \alpha^T\xi + \beta^T\eta = G(\xi, \eta, \zeta).$$

In view of (a), (ξ, η, ζ) is then optimal for D, and $\hat{F} = G(\xi, \eta, \zeta)$. An analogous reasoning, starting from D with $\hat{G} = G(\xi, \eta, \zeta)$ and using the fact that $\pi^{**} = \pi$, $\rho^{**} = \rho$, shows that P has an optimal solution (x, y, z) with $F(x, y, z) = \hat{G}$. Thus P and D have optimal solutions with $F = G$, and (1) is seen to be a necessary optimality condition. Its sufficiency is obvious from (a).

(c) Suppose now, that P has strictly feasible points and no linear part in the objective function. Then P requires the minimization of $\pi(z)$ subject to

$$\Phi_0 \cdot z \in \Phi_1 \cdot K + \Phi_2 \cdot L + k, \tag{11}$$

[2] "Théorème de Farkas-Minkowski. — Soient $f(x)$, $g_1(x)$, $g_2(x), \ldots, g_m(x)$ des fonctions concaves définies dans $\mathbb{R}^n$, et soit un indice $q \leqslant m$ tel que les fonctions $g_i(x)$ $(q < i \leqslant m)$ soient linéaires affines. Si le système $g_i(x) \geqslant 0$ $(i = 1, 2, \ldots, m)$, $(x) > 0$ n' admet pas de solution $x \in \mathbb{R}^n$, et si le système $g_i(x) > 0$ $(1 \leqslant i \leqslant q)$, $g_i(x) \geqslant 0$ $(q < i \leqslant m)$ admet une solution, alors il existe des nombres $y_1, y_2, \ldots, y_m \geqslant 0$ non tous nuls tels que $f(x) + \sum_1^m y_i g_i(x) \leqslant 0$ pour tout $x \in \mathbb{R}^n$."

where,

$$K = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \middle| \begin{matrix} x \geq 0 \\ u \geq 0 \end{matrix} \right\}, \qquad L = \{ y \mid \rho(y) \leq 1 \}, \qquad k = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\},$$

$$\Phi_0 = \begin{pmatrix} O \\ I \end{pmatrix}, \qquad \Phi_1 = \begin{pmatrix} A & I \\ C & O \end{pmatrix}, \qquad \Phi_2 = \begin{pmatrix} B \\ D \end{pmatrix}.$$

$\Phi_1 \cdot K$, being a linear transform of a polyhedral cone, is closed. $\Phi_2 \cdot L$, being a linear transform of a compact set, is compact. The right-hand side of (11), being the sum of two compact sets and a closed set, is closed. The set of feasible z is the inverse image of this closed set under Φ_0, thus is again closed. On the nonempty closed set of feasible z the norm $\pi(z)$ assumes a minimum. Therefore P has an optimal solution (x, y, z). The same reasoning as under (b) with $\hat{F} = F(x, y, z)$ shows that D has an optimal solution (ξ, η, ζ) with $G(\xi, \eta, \zeta) = F(x, y, z)$.

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